RESTRICTED ALGEBRAS ON INVERSE SEMIGROUPS I, REPRESENTATION THEORY

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Abstract. The relation between representations and positive definite functions is a key concept in harmonic analysis on topological groups. Recently this relation has been studied on topological groupoids. This is the first in a series of papers in which we have investigated a similar relation on inverse semigroups. We use a new concept of "restricted" representations and study the restricted semigroup algebras and corresponding C^*-algebras.

1. Introduction.

A continuous complex valued function $u : G \longrightarrow \mathbb{C}$ on a locally compact Hausdorff group G is called **positive definite** if for all positive integers n and all $c_1, \ldots, c_n \in \mathbb{C}$, and $x_1, \ldots, x_n \in G$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \bar{c}_i c_j u(x_i^{-1}x_j) \geq 0.$$

Positive definite functions on G are automatically bounded [11]. When G is Abelian, the positive definite functions are just the Fourier-Stieltjes transforms of Radon measures on the Pontryagin dual $\hat{G}$ of G [10]. The significance of positive definite functions is their relation with representations of G. For each representation $\{\pi, \mathcal{H}_\pi\}$ of G and each (unit) vector $\xi \in \mathcal{H}_\pi$, the map

$$x \mapsto < \pi(x)\xi, \xi >$$

on G is positive definite. Conversely each positive definite function on G is of this form [11].

Piere Eymard in [10] used positive definite functions to introduce the Fourier and Fourier-Stieltjes algebras $A(G)$ and $B(G)$ of a (not necessarily Abelian) locally compact group G (see also [12]). If G is a locally compact Abelian group, $A(G)$ and

1991 *Mathematics Subject Classification.* 43A35, 43A20.

Key words and phrases. Fourier algebra, inverse semigroups, restricted semigroup algebra.

This research was supported by Grant 510-2090 of Shahid Behshti University.

$B(G)$ are the ranges of the Fourier and Fourier-Stieltjes transforms on $L^1(G)$ and $M(G)$, respectively. Dunkl and Ramirez defined a subalgebra $R(S)$ of the algebra WAP(S) for a Hausdorff locally compact commutative topological semigroup S in [9](see also [13],14]). For a locally compact Abelian group G, $R(G) = M(\hat{G})\hat{}$. In [15] A.T. Lau defined a subalgebra $F(S)$ of WAP(S) for a topological $*$-semigroup S and then showed that if S has an identity, then $F(S) =< P(S) >$, where $P(S)$ is the set of all bounded positive definite functions on S. When S is commutative, we have $F(S) \subseteq R(S)$. The authors studied the Fourier and Fourier Stieltjes algebras $A(S)$ and $B(S)$ of a unital topological foundation $*$-semigroup in [5]. Also different versions of $A(G)$ and $B(G)$ are introduced and studied for measured and topological groupoids G in [18], [19], [22], and [23].

This is the first in a series of papers whose ultimate goal is a theory of Fourier algebras on inverse semigroups [1], [2]. There are many technical difficulties when one tries to do things similar to the group case. The first major obstacle is that the regular representation of an inverse semigroup looses its connection with positive definite functions. To avoid this difficulty we decided to introduce a new concept of representations. The new objects are called restricted representations. The basic idea is that we require the homomorphism property of representations to hold only for those pairs of elements in the semigroup whose range and domain "match". This is quite similar to what is done in the context of groupoids, but the representation theory of groupoids is much more involved (in [2] we investigate the relation between representations of inverse semigroups and their associated groupoids).

This paper is organized as follows: In section 2 we introduce the concept of restricted representations for inverse semigroups. In section 3 we introduce the restricted semigroup algebra and study its properties. In the last section the restricted versions of full and reduced semigroup C^*-algebras are introduced and studied.

2. Preliminaries

All over this paper, S denotes a unital inverse semigroup with identity 1. Let us remind that an inverse semigroup S is a discrete semigroup such that for each $s \in S$ there is a unique element $s^* \in S$ such that

$$ss^*s = s, \quad s^*ss^* = s^*.$$

Then one can show that $s \mapsto s^*$ is an involution on S. The set E of idempotents of S consists of elements the form ss^*, $s \in S$. E is a commutative sub semigroup

of S. There is a natural order $\leq$ on E defined by $e \leq f$ if and only if $ef = e$. We refer the reader to [21] for more details.

A $*$-representation of S is a pair $\{\pi, \mathcal{H}_\pi\}$ consisting of a (possibly infinite dimensional) Hilbert space $\mathcal{H}_\pi$ and a map $\pi : S \to \mathcal{B}(\mathcal{H}_\pi)$ satisfying

$$\pi(xy) = \pi(x)\pi(y),\ \pi(x^*) = \pi(x)^* \quad (x, y \in S),$$

that is a $*$-semigroup homomorphism from S into the inverse semigroup of partial isometries on $\mathcal{H}_\pi$. We loosely refer to π as the representation and it should be understood that there is always a Hilbert space coming with π. Let $\Sigma = \Sigma(S)$ be the family of all $*$-representations π of S with

$$\|\pi\| := \sup_{x \in S} \|\pi(x)\| \leq 1.$$

For $1 \leq p < \infty$, $\ell^p(S)$ is the Banach space of all complex valued functions f on S satisfying

$$\|f\|_p := \big(\sum_{x \in S} |f(x)|^p\big)^{\frac{1}{p}} < \infty.$$

For $p = \infty$, $\ell^\infty(S)$ consists of those f with $\|f\|_\infty := sup_{x \in S}|f(x)| < \infty$. Recall that $\ell^1(S)$ is a Banach algebra with respect to the product

$$(f * g)(x) = \sum_{st = x} f(s)g(t) \quad (f, g \in \ell^1(S)),$$

and $\ell^2(S)$ is a Hilbert space with inner product

$$< f, g >= \sum_{x \in S} f(x)\overline{g(x)} \quad (f, g \in \ell^2(S)).$$

Let also put

$$\check{f}(x) = f(x^*),\ \tilde{f}(x) = \overline{f(x^*)},$$

for each $f \in \ell^p(S) \quad (1 \leq p \leq \infty)$. We say that f is **symmetric**, if $f = \tilde{f}$.

Next, following [16], we introduce the associated groupoid of an inverse semigroup S. Given $x, y \in S$, the **restricted product** of x, y is xy if $x^*x = yy^*$, and undefined, otherwise. The set S with its restricted product forms a groupoid [16,3.1.4] which is called the **associated groupoid** of S and we denote it by S_a. If we adjoin a zero element 0 to this groupoid, and put $0^* = 0$, we get an inverse semigroup S_r with the multiplication rule

$$x \bullet y = \begin{cases} xy & \text{if } \ x^*x = yy^* \\ 0 & \text{otherwise} \end{cases} \quad (x, y \in S \cup \{0\}),$$

which is called the **restricted semigroup** of S. A **restricted representation** $\{\pi, \mathcal{H}_\pi\}$ of S is a map $\pi : S \to \mathcal{B}(\mathcal{H}_\pi)$ such that $\pi(x^*) = \pi(x)^* \quad (x \in S)$ and

$$\pi(x)\pi(y) = \begin{cases} \pi(xy) & \text{if } x^*x = yy^* \\ 0 & \text{otherwise} \end{cases} \qquad (x, y \in S).$$

Let $\Sigma_r = \Sigma_r(S)$ be the family of all restricted representations π of S with $\|\pi\| = \sup_{x\in S} \|\pi(x)\| \leq 1$. It is not hard to guess that $\Sigma_r(S)$ should be related to $\Sigma(S_r)$. Let $\Sigma_0(S_r)$ be the set of all $\pi \in \Sigma(S_r)$ with $\pi(0) = 0$. Note that $\Sigma_0(S_r)$ contains all cyclic representations of S_r. Now it is clear that, via a canonical identification, $\Sigma_r(S) = \Sigma_0(S_r)$.

One of the central concepts in the analytic theory of inverse semigroups (as in [4], [5],[6], [9], [17]) is the **left regular representation** $\lambda : S \longrightarrow B(\ell^2(S))$ defined by

$$\lambda(x)\xi(y) = \begin{cases} \xi(x^*y) & \text{if } xx^* \geq yy^* \\ 0 & \text{otherwise} \end{cases} \qquad (\xi \in \ell^2(S), x, y \in S).$$

This lifts to a faithful representation of $\ell^1(S)$ [24]. We define the **restricted left regular representation** $\lambda_r : S \longrightarrow B(\ell^2(S))$ by

$$\lambda_r(x)\xi(y) = \begin{cases} \xi(x^*y) & \text{if } xx^* = yy^* \\ 0 & \text{otherwise} \end{cases} \qquad (\xi \in \ell^2(S), x, y \in S).$$

Let us check that $\lambda_r \in \Sigma_r(S)$. Given $x \in S$, and $\xi, \eta \in \ell^2(S)$ we have $\|\lambda_r(x)\xi\|_2^2 = \sum_{xx^*=yy^*} |\xi(x^*y)|^2 \leq \sum_{z\in S} |\xi(z)|^2 = \|\xi\|_2^2$, so $\|\lambda_r\| = \sup_{x\in S} \|\lambda_r(x)\| \leq 1$. Also,

$$\begin{aligned} < \lambda_r(x^*)\xi, \eta > &= \sum_{y\in S} \lambda_r(x^*)\xi(y)\overline{\eta(y)} \\ &= \sum_{x^*x=yy^*} \xi(xy)\overline{\eta(y)} \\ &= \sum_{xx^*=zz^*} \xi(z)\overline{\eta(x^*z)} =< \xi, \lambda_r(x)\eta > . \end{aligned}$$

The last two sums are equal, because, if $x^*x = yy^*$, then by taking $xy = z$, we have $x^*z = x^*xy = y$, and $zz^* = xyy^*x^* = xx^*$. On the other hand, if $xx^* = zz^*$, then by taking $x^*z = y$, we have, $yy^* = x^*zz^*x = x^*x$.

Finally for each $x, y, z \in S$, $\xi \in \ell^2(S)$,

$$\lambda_r(xy)\xi(z) = \begin{cases} \xi(y^*x^*z) & \text{if } xyy^*x^* = zz^* \\ 0 & \text{otherwise,} \end{cases}$$

where as

$$\lambda_r(x)\lambda_r(y)\xi(z) = \begin{cases} \xi(y^*x^*z) & \text{if } xx^* = zz^*,\ yy^* = x^*zz^*x \\ 0 & \text{otherwise,} \end{cases}$$

Now conditions $xx^* = zz^*$ and $yy^* = x^*zz^*x$ imply conditions $x^*x = yy^*$ and $xyy^*x^* = zz^*$, so $\lambda_r(x)\lambda_r(y)$ is equal to $\lambda_r(xy)$, if $xx^* = zz^*$, and is equal to 0, otherwise. Hence $\lambda_r \in \Sigma_r(S)$, as claimed.

Next, given ξ, η in $\ell^2(S)$ or $\ell^1(S)$, we define

$$(\xi \bullet \eta)(x) = \sum_{x^*x=yy^*} \xi(xy)\eta(y^*) \quad (x \in S).$$

Then it is easy to check that in both cases this is a convergent sum. Also we clearly have

$$< \lambda_r(x^*)\xi, \eta >= \xi \bullet \tilde{\eta}(x),$$

for each $x \in S$, and $\xi, \eta \in \ell^2(S)$. To see the relation between this new dot product with the original convolution product on $\ell^1(S)$, it is useful to note that for $\xi, \eta \in \ell^1(S)$, $\xi \bullet \eta$ could be equivalently presented as

$$\xi \bullet \eta(x) = \sum_{s} \sum_{st=x, x^*x=t^*t} \xi(s)\eta(t) \quad (x \in S).$$

This in particular shows that if $\xi, \eta \geq 0$, then $\|\xi \bullet \eta\|_1 \leq \|\xi * \eta\|_1$, something which fails in general.

Similarly one can define the **restricted right regular representation** ρ_r of S in $\ell^2(S)$ by

$$\rho_r(x)\xi(y) = \begin{cases} \xi(yx) & \text{if } xx^* = y^*y \\ 0 & \text{otherwise} \end{cases} \quad (\xi \in \ell^2(S), x, y \in S),$$

and observe that $\rho_r \in \Sigma_r(S)$ and

$$< \rho_r(x)\xi, \eta >= \tilde{\eta} \bullet \xi(x) \quad (x \in S, \xi, \eta \in \ell^2(S)).$$

Also we have $< \tilde{\rho}_r(\varphi)\xi, \eta >= \varphi \bullet (\check{\xi} \bullet \bar{\eta})(1)$, for each $\varphi \in \ell^1(S)$ and $\xi, \eta \in \ell^2(S)$, where 1 is the identity of S. Indeed

$$\begin{aligned}
< \tilde{\rho}_r(\varphi)\xi, \eta > &= \sum_{y\in S} (\tilde{\rho}_r(\varphi)\xi)(y)\bar{\eta}(y) = \sum_y \sum_z \varphi(z)(\rho_r(z)\xi)(y)\bar{\eta}(y) \\
&= \sum_z \varphi(z) < \rho_r(z)\xi, \eta >= \sum_z \varphi(z)(\tilde{\eta} \bullet \xi)(z) \\
&= \sum_z \varphi(z)(\tilde{\eta} \bullet \xi)\check{}(z^*) = \varphi \bullet (\tilde{\eta} \bullet \xi)\check{}(1) \\
&= \varphi \bullet (\check{\xi} \bullet \bar{\eta})(1).
\end{aligned}$$

These identities simplify further calculations. Therefore we are led to consider the algebra $\ell^1(S)$ with respect to the dot product $\bullet$ (instead of convolution product $*$). We devote the next section to the study this new algebra.

3. Reduced semigroup algebra

In this section we show that for an inverse semigroup S, $(\ell^1(S), \bullet, \tilde{})$ is a Banach $*$-algebra with an approximate identity.

Proposition 3.1. *The dot product is associative.*

Proof Let $\xi, \eta, \theta \in \ell^1(S)$ and $x \in S$, then

$$(\xi \bullet \eta) \bullet \theta(x) = \sum_{x^*x=yy^*} (\xi \bullet \eta)(xy)\theta(y^*)$$

If we put $z = xy$ then $x^*z = x^*xy = y$ and $zz^* = xyy^*x^* = xx^*$. Conversely if $xx^* = zz^*$ then putting $y = x^*z$ we get $xy = xx^*z = z$ and $yy^* == x^*zz^*x = x^*x$. Hence the above sum is equal to

$$\sum_{xx^*=zz^*} (\xi \bullet \eta)(z)\theta(z^*x) = \sum_{xx^*=zz^*} \sum_{z^*z=vv^*} \xi(zu)\eta(v^*)\theta(z^*x).$$

On the other hand

$$\begin{aligned}
\xi \bullet (\eta \bullet \theta)(x) &= \sum_{x^*x=yy^*} \xi(xy)(\eta \bullet \theta)(y^*) \\
&= \sum_{x^*x=yy^*} \sum_{yy^*=uu^*} \xi(xy)\eta(y^*u)\theta(u^*)
\end{aligned}$$

Given $x \in S$, assume that z, v satisfy $xx^* = zz^*$ and $z^*z = vv^*$. Then put $u = x^*z$ and $y = x^*zv$. We have $uu^* = x^*zz^*x = x^*x$ and

$$yy^* = x^*zvv^*z^*x = x^*zz^*zz^*x = x^*zz^*x = x^*x = uu^*.$$

Conversely, if u, y satisfy $x^*x = uu^*$ and $uu^* = yy^*$, then put $z = xu$ and $v = u^*y$. We have $zz^* = xuu^*x^* = xx^*$ and $vv^* = u^*yy^*u = u^*u$, and $z^*z = u^*x^*xu = u^*u$, so $vv^* = u^*u$. Hence the two double sums which represent $(\xi \bullet \eta) \bullet \theta(x)$ and $\xi \bullet (\eta \bullet \theta)(x)$ are indeed the same. □

Theorem 3.1. *Under the usual norm, $(\ell^1(S), \bullet, \tilde{})$ is a Banach $*$-algebra.*

Proof We need only to check that $(f \bullet g)\tilde{} = \tilde{g} \bullet \tilde{f}$ and $\|f \bullet g\|_1 \leq \|f\|_1 \|g\|_1$, for each $f, g \in \ell^1(S)$. Fix $f, g \in \ell^1(S)$. For each $x \in S$,

$$\begin{aligned} (f \bullet g)\tilde{}(x) &= \sum_{xx^*=yy^*} \bar{f}(x^*y)\bar{g}(y^*) = \sum_{xx^*=yy^*} \tilde{f}(y^*x)\tilde{g}(y) \\ &= \sum_{x^*x=zz^*} \tilde{f}(z^*)\tilde{g}(xz) = \tilde{g} \bullet \tilde{f}(x). \end{aligned}$$

Next for $s, x \in S$ put $J_{s,x} = \{t \in S : st = x, x^*x = t^*t\}$ and note that for each $s, x, y \in S$ with $x \neq y$ we have $J_{s,x} \cap J_{s,y} = \emptyset$. This justifies the last inequality of the following calculation

$$\begin{aligned} \|f \bullet g\|_1 &= \sum_x |\sum_s \sum_{t \in J_{s,x}} f(s)g(t)| \leq \sum_x \sum_s \sum_{t \in J_{s,x}} |f(s)||g(t)| \\ &= \sum_s |f(s)| \sum_x \sum_{t \in J_{s,x}} |g(t)| \leq \sum_s |f(s)| \sum_t |g(t)| = \|f\|_1 \|g\|_1. \end{aligned}$$

□

Remark 3.1. *One may define the dot product $\bullet$ with the sum running through all elements y satisfying $yy^* \leq x^*x$ and relate it to the classical left regular representation λ quite similar to what we did with λ_r, but then the dot product $\bullet$ won't be associative in general. On the other hand, there is no connection between the usual convolution product $*$ on $\ell^1(S)$ and representation λ. For these reasons, it seems that the restricted version is inevitable if one insists to keep the relation between left regular representation and the multiplication on the semigroup algebra. This relation is the key for our computations all over the paper.*

We denote the above Banach algebra with $\ell^1_r(S)$ and call it the **restricted semigroup algebra** of S. Each $\pi \in \Sigma_r(S)$ lifts to an *-representation $\tilde{\pi}$ of $\ell^1_r(S)$ via

$$\tilde{\pi}(f) = \textstyle\sum_{x \in S} f(x)\pi(x) \quad (f \in \ell^1_r(S)).$$

To see that $\tilde{\pi}$ is a $*$-representation of $\ell^1_r(S)$, let $f, g \in \ell^1_r(S)$ and note that

$$\begin{aligned}\tilde{\pi}(f \bullet g) &= \sum_{x\in S}(f \bullet g)(x)\pi(x)\\ &= \sum_{x\in S}\sum_{x^*x=yy^*} f(xy)g(y^*)\pi(x),\end{aligned}$$

where as

$$\begin{aligned}\tilde{\pi}(f)\tilde{\pi}(g) &= \sum_{s\in S}\tilde{\pi}(f)g(s)\pi(s)\\ &= \sum_{s\in S}\sum_{t\in S} f(t)g(s)\pi(t)\pi(s)\\ &= \sum_{s\in S}\sum_{t^*t=ss^*} f(t)g(s)\pi(ts).\end{aligned}$$

Now the last sums of these presentations are converted to each other via change of variables $x = ts, y = s^*$ and $t = xy, s = y^*$.

Note that $\ell^1_r(S)$ is not necessarily unital (even if S is unital). However we show that it contains a (not necessarily bounded) approximate identity consisting of finitely supported functions.

Lemma 3.1. *Given $y \in S$, $e \in E$,*

$$\delta_y \bullet \delta_e = \begin{cases}\delta_y & y^*y = e\\ 0 & \textit{otherwise}\end{cases} \quad \textit{and} \quad \delta_e \bullet \delta_y = \begin{cases}\delta_y & yy^* = e\\ 0 & \textit{otherwise.}\end{cases}$$

Proof We have $\delta_y \bullet \delta_e(z) = \sum\limits_{z^*z=tt^*} \delta_y(zt)\delta_e(t^*) = 0$, unless $t = e$ and $zt = y$ and $z^*z = tt^*$. If these equalities hold, then $z^*z = tt^* = t$, so $y = zt = zz^*z = z$. Therefore $(\delta_y \bullet \delta_e)(z) = 0$, unless $z = y$. Now $(\delta_y \bullet \delta_e)(y) = \sum_{y^*y=tt^*} \delta_y(yt)\delta_e(t^*) = 0$, unless $t = e$, $yt = y$, and $y^*y = tt^*$. These equalities imply that $y^*y = tt^* = e$. Conversely if $y^*y = e$, then

$$\delta_y \bullet \delta_e(y) = \sum_{y^*y=tt^*} \delta_y(yt)\delta_e(t^*) = \delta_y(ye) = \delta_y(y) = 1.$$

The other statement is proved similarly. $\square$

Now for each finite subset $F = \{x_1, \dots, x_n\}$ of S let us put

$$i(F) = \{e \in E : e = xx^* \text{ or } x^*x, \text{ for some } x \in F\},$$

which is clearly a finite subset of E. Define

$$e_F = \sum_{e\in i(F)} \delta_e.$$

Lemma 3.2. *Let F, G be finite subsets of S and e_F, e_G be as above, then*

(i) For each $F_0 \subseteq F$ and each $s \in F_0$, $e_F \bullet \delta_s = \delta_s \bullet e_F = \delta_s$,

(ii) $e_F \bullet e_G = e_G \bullet e_F = \sum_{e\in i(F)\cap i(G)} \delta_e$, in particular if $G \subseteq F$, then $e_F \bullet e_G = e_G \bullet e_F = e_G$.

(iii) For each $f = \sum_{i=1}^{\infty} f(s_i)\delta_{s_i} \in \ell_r^1(S)$,

$$f \bullet e_F = \sum_{s_i^* s_i \in i(F)} f(s_i)\delta_{s_i},$$

and

$$e_F \bullet f = \sum_{s_i s_i^* \in i(F)} f(s_i)\delta_{s_i}.$$

(iv) If $f \in \ell_r^1(S)$ and $supp(f) \subseteq F$, then $f \bullet e_F = e_F \bullet f = f$.

Proof For each $s \in F_0$, there are unique $e, f \in i(F)$ such that $s^*s = e$ and $ss^* = f$. By Lemma 3.1, for each $g \in i(F)$, $\delta_s \bullet \delta_g$ is equal to δ_s, if $g = e$, and is 0, otherwise. A similar statement, with f replaced by e, holds for the multiplication by δ_s from right. This shows (i). Also the above lemma shows that for each $e, f \in E$, $\delta_e \bullet \delta_f = \delta_f \bullet \delta_e$ is δ_e, if $e = f$, and is 0, otherwise. Hence

$$e_F \bullet e_G = \sum_{e\in i(F), f\in i(G)} \delta_e \bullet \delta_f = \sum_{e\in i(F)\cap i(G)} \delta_e,$$

which proves the first statement of (ii). In particular if $G \subseteq F$, we get $e_F \bullet e_G = e_G \bullet e_F = e_G$. Next let $f \in \ell_r^1(S)$ and choose any $s_i \in supp(f)$. If $s_i^* s_i \notin i(F)$, then by above lemma, $\delta_{s_i} \bullet \delta_e = 0$, for each $e \in i(F)$, and so $\delta_{s_i} \bullet e_F = 0$. If $s_i^* s_i \in i(F)$, then there is a unique $e \in i(F)$ such that $s_i^* s_i = e$, so again by above lemma, $\delta_{s_i} \bullet e_F = \delta_{s_i} \bullet \delta_e = \delta_{s_i}$. This proves the first equality in (iii). The proof of the second equality is similar. Finally (iv) follows from (iii). $\square$

Proposition 3.2. *The Banach algebra $\ell_r^1(S)$ has a (not necessarily bounded) two sided approximate identity consisting of positive, symmetric functions of finite support.*

Proof For each finite subset F of S let e_F be as above. It is clear that e_F is a positive, symmetric function of finite support. Given $f \in l_r^1(S)$ we have $\sum\limits_{s\in S} |f(s)| < \infty$, so there are at most countably many $s \in S$, say $s_1, s_2, \ldots$, for which $f(s) \neq 0$. Then given $\epsilon > 0$, there is $N \geq 1$ s.t. $\sum_{i=N+1}^{\infty} |f(s_i)| < \epsilon$. Put $F_0 = \{s_1, \ldots, s_N\}$

and let $F \supseteq F_0$, then by the first equality in part (iii) of the above lemma,

$$\|f - f \bullet e_F\|_1 = \sum_{s_i^* s_i \notin i(F)} |f(s_i)| \leq \sum_{s_i^* s_i \notin i(F_0)} |f(s_i)|$$
$$\leq \sum_{i \geq N+1} |f(s_i)| < \epsilon.$$

Similarly we have $\|f - e_F \bullet f\|_1 < \epsilon$. □

Note that with f, F_0 and F as above, a similar argument could show that $\|f - e_F \bullet f \bullet e_F\|_1 < \epsilon$. The above result looks more interesting when one recalls that if S is not unital, $\ell^1(S)$ may fail to have a bounded (or even unbounded) approximate identity. This could be the case for $\ell^1(S_r)$, but, in the light of the next proposition, what we have shown is that $\ell^1(S_r)$ has an approximate identity, provided that we identify two elements of $\ell^1(S_r)$ which agree at 0.

Proposition 3.3. *The restriction map $\tau : \ell^1(S_r) \to \ell^1_r(S)$ is a surjective contractive Banach algebra homomorphism whose kernel is $\mathbb{C}\delta_0$. The quotient Banach algebra $\ell^1(S_r)/\mathbb{C}\delta_0$ is isometrically isomorphic to $\ell^1_r(S)$.*

Proof Recall that $S_r = S^0$ is a semigroup with respect to the restricted product. Let us denote the convolution product on $\ell^1(S_r)$ by $\tilde{*}$, then

$$\delta_x \tilde{*} \delta_y = \delta_{x \bullet y} = \delta_x \bullet \delta_y \quad (x, y \in S_r),$$

where the second equality is trivial if $x = 0$ or $y = 0$, and could be easily checked (similar to the proof of Lemma 3.1) when $x, y \in S$. This shows that τ is a homomorphism. All the other assertions are trivial, except that we have to check $\|\tau(f)\|_1 = \|f + \mathbb{C}\delta_0\|$, for each $f \in \ell^1(S_r)$. But the right hand side is the infimum over all $c \in \mathbb{C}$ of $\|f + c\delta_0\|_1$, which is clearly obtained at $c = -f(0)$, since $\|f + c\delta_0\|_1 = \|\sum_s f(s)\delta_s + (c + f(0))\delta_0\|_1 = \sum_s |f(s)| + |c + f(0)|$. □

4. RESTRICTED SEMIGROUP C^*-ALGEBRAS

We know that when S is an inverse semigroup (not necessarily unital), the left regular representation λ lifts to a faithful representation $\tilde{\lambda}$ of $\ell^1(S)$ [24]. In particular $\|f\|_\lambda = \|\tilde{\lambda}(f)\|$ defines a C^*-norm on $\ell^1(S)$. The completion of $\ell^1(S)$ in this norm is called the **reduced C^*-algebra** of S and is denoted by $C^*_\lambda(S)$. Now let $\Sigma = \Sigma(S)$ and for each $f \in \ell^1(S)$ define

$$\|f\|_\Sigma = \sup\{\|\tilde{\pi}(f)\| : \pi \in \Sigma = \Sigma(S)\},$$

where

$$\tilde{\pi}(f) = \sum_{x \in S} f(x)\pi(x) \quad (f \in \ell^1(S)).$$

Then for each irreducible representation $\theta : C^*_\lambda(S) \longrightarrow B(\mathcal{H}_\theta)$, $\tilde{\pi} = \theta \circ \tilde{\lambda}$ is an irreducible representation of $\ell^1(S)$ and so $\|f\|_\lambda \leq \|f\|_\Sigma$ $(f \in \ell^1(S_r))$ [9]. In particular $\|\cdot\|_\Sigma$ is a C^*-norm. This is the largest C^*-norm on $\ell^1(S)$ and the corresponding enveloping C^*-algebra is denoted by $C^*(S)$ and is called the **(full) C^*-algebra** of S [8]. (For full and reduced C^*-algebras on topological groups see [7], [20], for topological semigroups see [3],[15]). Also $\tilde{\lambda}$ extends uniquely to an *-epimorphism $\tilde{\lambda} : C^*(S) \longrightarrow C^*_\lambda(S)$.

Now let us consider unital inverse semigroup S and its associated inverse 0-semigroup S_r. In this section we want to explore the same ideas for $\ell^1_r(S)$. As all of the above results are valid for the non unital inverse semigroups, we can freely apply them to S_r.

Recall that $\Sigma_r(S) = \Sigma_0(S_r)$. In particular, the restricted left regular representation λ_r of S corresponds to a representation of S_r which vanishes at 0. Indeed let Λ dente the left regular representation of S_r. Consider the closed subspace $\ell^2_0(S_r) := \{\xi \in \ell^2(S_r) : \xi(0) = 0\}$ of $\ell^2(S_r)$, and let $P_0 : \ell^2(S_r) \to \ell^2_0(S_r)$, $\xi \mapsto \xi - \xi(0)\delta_0$ be the corresponding orthogonal projection, then the restriction map is an isomorphism of Hilbert spaces from $\ell^2_0(S_r)$ onto $\ell^2(S)$, and under this identification, $\lambda_r(s) = \Lambda(s)P_0$. By the Wordingham's theorem [24] we know that $\tilde{\Lambda}$ is a faithful representation of $\ell^1(S)$. Unfortunately the relation $\tilde{\lambda}_r = \tilde{\Lambda}(.)P_0$ does not necessarily imply that $\tilde{\lambda}_r$ is also faithful, but a rather straightforward argument (even much easier than that of [24]) shows that $\tilde{\lambda}_r$ is faithful.

Lemma 4.1. *$\tilde{\lambda}_r$ is faithful.*

Proof Fix any $f \in \ell^1_r(S)$ with $\tilde{\lambda}_r(f) = 0$. Let $u \in S$ and put $t = uu^*$ and $\xi = \delta_{u^*} \in \ell^2(S)$, then

$$\begin{aligned} 0 = \tilde{\lambda}_r(f)\xi(t) &= \sum_{s \in S} f(s)(\lambda_r(s)\delta_{u^*})(uu^*) \\ &= \sum_{ss^* = uu^*} f(s)\delta_{u^*}(s^*uu^*) \\ &= \sum_{ss^*=uu^*, s^*uu^*=u^*} f(s) = f(u), \end{aligned}$$

so $f = 0$. □

Corollary 4.1. *The Banach algebra $\ell^1_r(S)$ is semi-simple.* □

The above lemma shows that $\|f\|_{\lambda_r} := \|\tilde{\lambda}_r(f)\|$ defines a C^*-norm on $\ell^1_r(S)$. We call the completion $C^*_{\lambda_r}(S)$ of $\ell^1_r(S)$ in this norm, the **restricted reduced C^*-algebra** of S. Next consider

$$\|f\|_{\Sigma_r} = \sup\{\|\tilde{\pi}(f)\| : \pi \in \Sigma_r = \Sigma_r(S)\} \quad (f \in \ell^1_r(S)),$$

then clearly $\|f\|_{\lambda_r} \leq \|f\|_{\Sigma_r}$ and so $\|\cdot\|_{\Sigma_r}$ is also a C^*-norm. Since $\tilde{\pi}$'s with $\pi \in \Sigma_r(S)$ exhaust all the non degenerate $*$-representations of $\ell^1_r(S)$, this is indeed the largest C^*-norm on $\ell^1_r(S)$. We call the completion $C^*_r(S)$ of $\ell^1_r(S)$ in this norm, the **restricted full C^*-algebra** of S. As in the classical case, $\tilde{\lambda}_r$ extends uniquely to an *-epimorphism $\tilde{\lambda}_r : C^*_r(S) \longrightarrow C^*_{\lambda_r}(S)$.

Next we find the relation between the restricted full and reduced C^*-algebras of S with the full and reduced C^*-algebras of S_r. The following technical lemma is probably true in a more general form.

Lemma 4.2. *Let $(A, \|.\|_1)$ be a Banach algebra and $\|.\|$ be a C^*-norm on A satisfying $\|.\| \leq \|.\|_1$. Let $C^*(A)$ be the completion of A in $\|.\|$. Let J be a two sided ideal of A which is closed in $(C^*(A), \|.\|)$. Then the quotient norm on $C^*(A)/J$ induces a C^*-norm on A/J and the C^*-completion $C^*(A/J)$ of A/J in this norm is isometrically isomorphic to $C^*(A)/J$.*

Proof J is clearly a closed ideal of $(A, \|.\|_1)$. Hence A/J is a Banach algebra under the quotient norm induced by $\|.\|_1$ which is clearly dense in $C^*(A)/J$ in its C^*-quotient norm. $\square$

We apply the above lemma to $A = \ell^1(S_r)$, $J = \mathbb{C}\delta_0$, and C^*-norms $\|.\|_\Lambda$ and $\|.\|_{\Sigma(S_r)}$. Note that $\mathbb{C}\delta_0$ is closed in both of the above C^*-norms. Indeed, given $c \in \mathbb{C}$, we have

$$\|c\delta_0\|_{\sigma(S_r)} = sup_{\pi\in\Sigma(S_r)}\|\tilde{\pi}(c\delta_0)\| = sup_{\pi\in\Sigma(S_r)}\|c\pi(0)\| = |c|,$$

and

$$\|c\delta_0\|_\Lambda = \|\tilde{\Lambda}(c\delta_0)\| = \|c\Lambda(0)\| = |c|,$$

where the last equality follows from the fact that $\|\Lambda(0)\| = 1$ (inequality in one direction is already known, for the other direction use $\Lambda(0)\delta_0 = \delta(0)$). Therefore any net $\{c_\alpha\delta_0\}$ which is convergent in any of the above norms would result in a Cauchy net $\{c_\alpha\}$ in $\mathbb{C}$. If $c_\alpha \to c$ in $\mathbb{C}$, then the given net would converge to $c\delta_0$ in both norms.

Therefore, in the light of Proposition 3.3, The following result follows immediately from the above lemma.

Proposition 4.1. *We have the isometric isomorphisms of C^*-algebras* $C^*_{\lambda_r}(S) \simeq C^*_\Lambda(S_r)/\mathbb{C}\delta_0$ *and* $C^*_r(S) \simeq C^*(S_r)/\mathbb{C}\delta_0$. □

acknowledgement. We would like to thank Dr. David Cowan who pointed out an error in the first version of this paper. The first author would like to thank hospitality of Professor Mahmood Khoshkam during his stay in University of Saskatchewan, where the main part of the revision was done.

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